

Multidimensional effective S-adic systems are sofic.

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Abstract. In this article we prove that multidimensional effective S-adic systems, obtained by applying an effective sequence of substitutions chosen among a finite set of substitutions, are sofic subshifts.

Introduction

Let \mathcal{A} be a finite alphabet. A d -dimensional subshift $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ is a closed and shift-invariant set of configurations, where the shift is the natural action of \mathbb{Z}^d on the configurations space $\mathcal{A}^{\mathbb{Z}^d}$. With a combinatorial point of view, one can equivalently define subshifts by excluding configurations that contain some forbidden finite patterns. Depending on the conditions imposed on this set of forbidden patterns, it is possible to define several classes of subshifts. The simplest one is the class of subshifts of finite type (also called SFT), where the set of forbidden finite patterns may be chosen finite. A larger class is the one of sofic subshifts, which are images of SFT under a factor map. These two classes are defined locally and they are well understood in dimension 1.

A way to construct minimal aperiodic subshifts is to consider subshifts generated by a fix point of substitution, introduced in dimension one by Thue [Thu06] and generalized to higher dimensions. These subshifts constitute the class of the substitutive subshifts. More precisely, for a substitution s one can consider the subshift $\mathbf{T}_{\{s\}}$ where the allowed patterns are given by iterations of the substitution s on a letter of \mathcal{A} , or $\mathbf{T}'_{\{s\}}$ the set of configurations which have pre-images by arbitrarily many iterations of s . Of course $\mathbf{T}_{\{s\}} \subset \mathbf{T}'_{\{s\}}$. In dimension 1 the class of substitutive subshifts and the class of sofic subshifts are disjoint: substitutive subshifts have low complexity [Pan84], while the only sofic subshifts with low complexity are periodic. In the multidimensional framework the situation is different since all substitutive subshifts are sofic. This result is a generalization to any substitution [Moz89] of the original construction of aperiodic tilings [Rob71].

A possible generalization of the construction of substitutive subshifts is to consider S-adic subshifts, that were introduced by S. Ferenczi in the one-dimensional setting [Fer96]. Given a finite set of substitutions \mathcal{S} , and a sequence $S \in \mathcal{S}^{\mathbb{N}}$, we define the subshifts \mathbf{T}_S and \mathbf{T}'_S where the iterations of the different substitutions are given by the sequence S . This class of subshifts is studied in dimension 1 and, under some condition on the set \mathcal{S} , it is shown that the complexity is low [Fer96, Dur00]. It is thus natural to wonder if there exists sofic S-adic subshifts in higher dimensions. An argument of cardinality shows that the class of

S-adic multidimensional subshifts is not included in the class of sofic subshifts. Indeed the class of sofic subshifts is countable while there are uncountably many ways to choose an infinite sequence of \mathcal{S} . The purpose of this article is to show that S-adic subshifts which are sofic are exactly those for which the sequence S is effective.

The main idea of the proof is to use the result of S. Mozes [Moz89] which proves that a substitutive subshift is sofic in the case where the substitution is not deterministic. This means that each time one wants to use a substitution, it is possible to choose a rule among a set of substitutions \mathcal{S} . However, contrary to the S-adic subshifts, at each level of iteration different substitutions of \mathcal{S} may appear. The aim of the proof is to synchronize these substitutions, and in that purpose we need to introduce a one dimensional effective subshift which codes the sequence of substitutions. This effective subshift can be realized by a 3-dimensional sofic subshift thanks to the result of M. Hochman [Hoc09] or by a 2-dimensional sofic subshift thanks to the improvement obtained by [DRS09] or [AS10].

1 Definition and classical properties

1.1 Notion of subshift

Let \mathcal{A} be a finite alphabet and d be a positive integer. A *configuration* x is an element of $\mathcal{A}^{\mathbb{Z}^d}$. Let \mathbb{U} be a finite subset of \mathbb{Z}^d , we denote by $x_{\mathbb{U}}$ the *restriction* of x to \mathbb{U} . A \mathbb{Z}^d -*dimensional pattern* is an element $p \in \mathcal{A}^{\mathbb{U}}$ where $\mathbb{U} \subset \mathbb{Z}^d$ is finite, \mathbb{U} is the *support* of p , which is denoted by $\text{supp}(p)$. A pattern p of support $\mathbb{U} \subset \mathbb{Z}^d$ *appears* in a configuration x if there exists $i \in \mathbb{Z}^d$ such that $p_{\mathbb{U}} = x_{i+\mathbb{U}}$, and in this case we note $p \sqsubset x$.

We define a topology on $\mathcal{A}^{\mathbb{Z}^d}$ by endowing \mathcal{A} with the discrete topology, and considering the product topology on $\mathcal{A}^{\mathbb{Z}^d}$. For this topology, $\mathcal{A}^{\mathbb{Z}^d}$ is a compact metric space on which \mathbb{Z}^d acts by translation via σ defined for every $i \in \mathbb{Z}^d$ by:

$$\sigma^i_{\mathcal{A}} : \left(\begin{array}{ccc} \mathcal{A}^{\mathbb{Z}^d} & \longrightarrow & \mathcal{A}^{\mathbb{Z}^d} \\ x & \longmapsto & \sigma^i_{\mathcal{A}}(x) \end{array} \text{ such that } \sigma^i_{\mathcal{A}}(x)_u = x_{i+u} \ \forall u \in \mathbb{Z}^d \right).$$

The \mathbb{Z}^d -action $(\mathcal{A}^{\mathbb{Z}^d}, \sigma)$ is called the *fullshift*. Let $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ be a closed subset σ -invariant, the \mathbb{Z}^d -action (\mathbf{T}, σ) is a *subshift*.

Let F be a set of finite patterns, we define the *subshift of forbidden patterns* F by

$$\mathbf{T}_F = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall p \in F, p \text{ does not appear in } x \right\}.$$

It is well known that every subshift can be defined by this way [LM95]. Let \mathbf{T} be a subshift. If there exists a finite set of forbidden patterns such that $\mathbf{T} = \mathbf{T}_F$, then \mathbf{T} is a *subshift of finite type*. If there exists a recursively enumerable set of forbidden patterns such that $\mathbf{T} = \mathbf{T}_F$, then \mathbf{T} is an *effective subshift*.

1.2 Factor and projective subaction

Let $(\mathcal{A}^{\mathbb{Z}^d}, \sigma)$ and $(\mathcal{B}^{\mathbb{Z}^d}, \sigma)$ be two fullshifts. A *factor* is a continuous function $\pi : \mathcal{A}^{\mathbb{Z}^d} \rightarrow \mathcal{B}^{\mathbb{Z}^d}$ such that $\pi \circ \sigma = \sigma \circ \pi$. Let $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ be an SFT and let $\pi : \mathcal{A}^{\mathbb{Z}^d} \rightarrow \mathcal{B}^{\mathbb{Z}^d}$ be a factor, then $\pi(\mathbf{T}) \subset \mathcal{B}^{\mathbb{Z}^d}$ is a subshift called a *sofic subshift*. In dimension 1, sofic subshifts are well understood since they are exactly subshifts whose language is regular [LM95].

Let \mathbb{G} be a sub-group of \mathbb{Z}^d finitely and freely generated by $u_1, u_2, \dots, u_{d'}$ ($d' \leq d$). Let $\mathbf{T} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift, the *projective subdynamic* – or *projective subaction* – of \mathbf{T} according to \mathbb{G} is the subshift of dimension d' defined by

$$\mathbf{SA}_{\mathbb{G}}(\mathbf{T}) = \left\{ y \in \mathcal{A}^{\mathbb{Z}^{d'}} : \exists x \in \mathbf{T} \text{ such that } \forall i_1, \dots, i_{d'} \in \mathbb{Z}^{d'}, y_{i_1, \dots, i_{d'}} = x_{i_1 u_1 + \dots + i_{d'} u_{d'}} \right\}.$$

In [PS10] the authors show that any 1-dimensional sofic subshift with positive entropy can be obtained as the projective subaction of a 2-dimensional SFT, but it remains open to know whether these subshifts are the only ones that can be obtained in that way. A complete characterization was obtained by Hochman [Hoc09] if we allow factor maps in addition to projective subactions: the class of subshifts obtained by factor maps and projective subactions of SFT is exactly the class of effective subshifts. The original proof contains a construction that realizes any 1-dimensional effective subshift inside a 3-dimensional SFT. This construction was improved simultaneously by two different techniques [AS10, DRS10] to get any 1-dimensional effective subshift inside a 2-dimensional SFT.

Theorem 1 ([Hoc09, AS10, DRS10]). *Any effective subshift of dimension d can be obtained with factor and projective subaction operations from a subshift of finite type of strictly higher dimension.*

2 Substitutive and S-adic subshifts

In this section we present substitutives and S-adic subshifts and prove that multidimensional S-adic subshifts given by an effective sequence of substitutions are sofic.

2.1 Substitutions

Let $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, we define $\mathbf{n} + \mathbf{k} = (n_1 + k_1, \dots, n_d + k_d) \in \mathbb{N}^d$, $\mathbf{n} \otimes \mathbf{k} = (n_1.k_1, \dots, n_d.k_d) \in \mathbb{N}^d$ and $\mathbf{n}^i = \mathbf{n} \otimes \dots \otimes \mathbf{n}$ with i factors. Given $\mathbf{k} = (k_1, \dots, k_d)$, we denote by $\mathbb{U}_{\mathbf{k}}$ the rectangle $[0; k_1] \times [0; k_2] \times \dots \times [0; k_d]$. We say that \mathbf{i} is smaller (resp. strictly smaller) than \mathbf{j} if for every $1 \leq l \leq d$, one has $i_l \leq j_l$ (resp. $i_l < j_l$). We denote it by $\mathbf{i} \leq \mathbf{j}$ (resp. $\mathbf{i} < \mathbf{j}$).

Let \mathcal{A} be a finite alphabet, we define the *set of rectangular pattern* $\mathcal{P} = \bigcup_{\mathbf{k} \in \mathbb{N}^d} \mathcal{A}^{\mathbb{U}_{\mathbf{k}}}$. A (\mathcal{A}, d) -*multidimensional substitution* of size \mathbf{k} is a function $s :$

$\mathcal{A} \rightarrow \mathcal{P}$, for all $a \in \mathcal{A}$, we associate the vector $\mathbf{k}^s(a) = (k_1^s(a), \dots, k_d^s(a))$ such that $\text{supp}(s(a)) = \mathbb{U}_{\mathbf{k}^s(a)}$, that is to say the support of $s(a)$ depends on the letter a . A (\mathcal{A}, d) -multidimensional substitution is non degenerate if $k_l^s(a) \geq 1$ for every $l \in [1, d]$ and every $a \in \mathcal{A}$.

Let $(\mathbf{k}^n)_{n \in \mathbb{Z}}$ be a sequence of d -dimensional vectors. For each $n \in \mathbb{Z}$ we define the function

$$\phi^{(\mathbf{k}^n)_{n \in \mathbb{Z}}} : \left(\begin{array}{c} \mathbb{Z}^d \rightarrow \mathbb{Z}^d \\ \mathbf{i} \mapsto (\phi_1(\mathbf{i}), \phi_2(\mathbf{i}), \dots, \phi_d(\mathbf{i})) \end{array} \right)$$

where $\phi_i(r) = \sum_{j=0}^r (\mathbf{k}^j)_i$ if $r \geq 0$ and $\phi_i(r) = \sum_{j=-1}^r (\mathbf{k}^j)_i$ if $r < 0$.

Let $p \in \mathcal{A}^{\mathbb{U}_{\mathbf{k}}}$ be a rectangular pattern with finite support $\mathbb{U}_{\mathbf{k}} \subset \mathbb{Z}^d$. We say that the substitution s is *compatible* with the pattern p (resp. the configuration x) if for all $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{U}$ and $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{U}$ (resp. $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$ and $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}^d$) such that $i_l = j_l$ for one $l \in [1, d]$, one has $k_l^s(p_{\mathbf{i}}) = k_l^s(p_{\mathbf{j}})$. Given a substitution s compatible with a configuration $x \in \mathbb{Z}^d$, one can transform \mathbb{Z}^d into a non-regular grid thanks to the function $\phi^x = \phi^{(\mathbf{k}^s(x_{(n, \dots, n)}))_{n \in \mathbb{Z}}}$ (see Figure1).

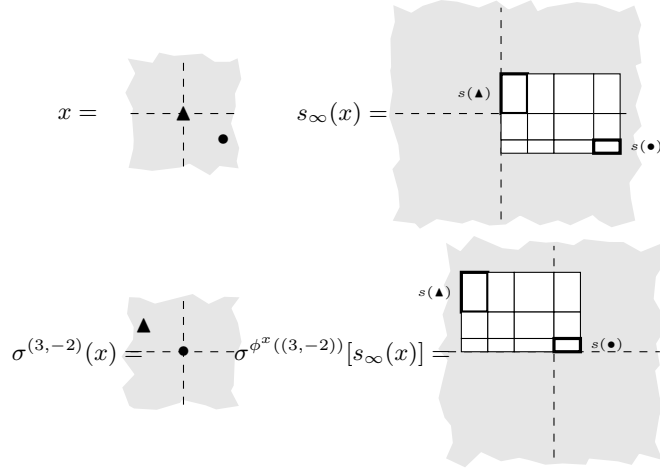


Fig. 1. If the configuration x is compatible with the substitution s , then one can define $\phi^x = \phi^{(\mathbf{k}^s(x_{(n, \dots, n)}))_{n \in \mathbb{Z}}}$.

If the substitution s is compatible with the configuration x that contains a pattern p , the substitution s acts on p and we obtain a pattern $s(p)$ whose support is

$$\text{supp}(s(p)) = \bigcup_{\mathbf{i} \in \text{supp}(p)} \mathbb{U}_{\mathbf{k}^s(p_{\mathbf{i}})} + \phi^x(\mathbf{i})$$

and such that

$$\forall \mathbf{i} \in \text{supp}(p), \forall \mathbf{j} \in \text{supp}(s(p_{\mathbf{i}})), s(p)_{\phi^x(\mathbf{i})+\mathbf{j}} = s(p_{\mathbf{i}})_{\mathbf{j}}.$$

So the substitution s can easily be extended to a function on configurations $s_{\infty} : \left(\begin{array}{c} \mathcal{A}^{\mathbb{Z}^d} \rightarrow \mathcal{A}^{\mathbb{Z}^d} \\ x \mapsto s(x) \end{array} \right)$ where if the substitution s is compatible with the configuration $x \in \mathcal{A}^{\mathbb{Z}^2}$, the configuration $s_{\infty}(x)$ is defined as explained above.

Example 1. Let \mathcal{A} be the two elements alphabet $\mathcal{A} = \{\circ, \bullet\}$ and s be the two-dimensional substitution whose rules are

$$\circ \mapsto \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \text{ and } (\bullet, \circ) \mapsto \begin{array}{cc} \circ & \circ \\ \bullet & \circ \end{array}.$$

For instance, the substitution s acts on the pattern p described below

$$s : p = \begin{array}{cc} \circ & \bullet \\ \bullet & \circ \end{array} \mapsto s(p) = \frac{\begin{array}{cc|cc} \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{array}}{\begin{array}{cc|cc} \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{array}}.$$

2.2 Composition of substitutions

Consider now that one wants to apply not only one but a finite set of substitutions on a (finite or not) pattern p . We first define how to compute the composition of two or more substitutions. Let s, s' be two substitutions. We say that s' is compatible with s if for any pattern p compatible with s , $s(p)$ is compatible with s' . If s' is compatible with s , we can thus define the composition $s' \circ s$. For a sequence of substitutions $S_{[0,n]} = (s_0, \dots, s_n)$, one defines by induction the substitution $\widehat{S_{[0,n]}}$ if s_0 is compatible with $\widehat{S_{[1,n]}}$ by $\widehat{S_{[0,n]}} = s_0 \circ \widehat{S_{[1,n]}}$.

Let \mathcal{S} be a finite set of (\mathcal{A}, d) -multidimensional substitutions. We present the two classical points of view to make \mathcal{S} act on the set of configurations $\mathcal{A}^{\mathbb{Z}^d}$. In the first one, the set \mathcal{S} acts on a configuration x via a sequence of substitutions $S = (s_i)_{i \in \mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$, and at iteration i the substitution s_i is applied to every letter in x (see Section 2.3). In the second one, the set \mathcal{S} acts on a configuration x in a non uniform way, that is to say at each iteration the substitution applied depends on the position in x (see Section 2.4).

2.3 S-adic subshifts

Let \mathcal{S} be a finite set of (\mathcal{A}, d) -multidimensional substitutions and let $S \in \mathcal{S}^{\mathbb{N}}$ be a sequence of substitutions. We want to define how this sequence acts on a letter $a \in \mathcal{A}$. The principle is that at the iteration number i , the substitution s_0 is applied to the whole pattern $s_1 \circ \dots \circ s_i(a)$. We define the two S -adic subshifts

based on this action of S on letters of \mathcal{A}

$$\begin{aligned}\mathbf{T}_S &= \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall p \sqsubset x, \exists a \in \mathcal{A}, \exists n \in \mathbb{N}, \quad p \sqsubset \widehat{S_{[0,n]}}(a) \right\} \\ \mathbf{T}'_S &= \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall n \in \mathbb{N}, \exists y \in \mathcal{A}^{\mathbb{Z}^d}, \quad \widehat{S_{[0,n]}}(y) = x \right\}.\end{aligned}$$

The first subshift \mathbf{T}_S will be called the *local S -adic subshift*. With the sequence of substitutions S we can produce patterns, that are the $\widehat{S_{[0,n]}}(a)$ for any letter $a \in \mathcal{A}$, and these patterns are seen as the allowed pattern of the subshift \mathbf{T}_S . The second subshift \mathbf{T}'_S will be called the *global S -adic subshift*, and this time the idea is to consider only configurations $x \in \mathcal{A}^{\mathbb{Z}^2}$ for which it is possible to find a pre-image of any order under the sequence of substitutions S . Obviously $\mathbf{T}_S \subset \mathbf{T}'_S$, but the reciprocal inclusion does not necessary hold as Example 2 shows.

Example 2. Let s be the substitution of Example 1. Then if we choose $\mathcal{S} = \{s\}$ and so $S = s^{\mathbb{N}}$, the two S -adic subshifts defined above are in this case

$$\mathbf{T}_S = \left\{ \circ^{\mathbb{Z}^2} \right\} \quad \text{and} \quad \mathbf{T}'_S = \left\{ \circ^{\mathbb{Z}^2} \right\} \cup \left\{ \sigma^i(x_\bullet), i \in \mathbb{Z}^2 \right\}$$

where the configuration x_\bullet such that $x_{(i,j)} = \circ$ if and only if $i \neq 0$ and $j \neq 0$ is in the subshift \mathbf{T}'_S — x_\bullet is a fixed-point of s — but not in the subshift \mathbf{T}_S , since the central pattern $x_{\mathbb{U}_1}$ appears neither in a $\widehat{S_{[0,n]}}(\bullet)$ nor in a $\widehat{S_{[0,n]}}(\circ)$.

2.4 Non-deterministic substitutions

Let \mathcal{S} be a set of substitutions, another way to consider the action of this set on a configuration in a non-deterministic (that is to say non-uniform) way. That is to say, if we consider a pattern $\mathcal{A}^{\mathbb{U}}$, each letter can be modified by a distinct substitution.

For a finite set $\mathbb{U} \subset \mathbb{Z}^d$, we consider the pattern of substitutions $\mathbf{s} \in \mathcal{S}^{\mathbb{U}}$. We say that the pattern of substitution $\mathbf{s} \in \mathcal{S}^{\mathbb{U}}$ is *compatible* with a pattern $p \in \mathcal{A}^{\mathbb{U}}$ if for all $i = (i_1, \dots, i_d) \in \mathbb{U}$ and $j = (j_1, \dots, j_d) \in \mathbb{U}$ such that $i_l = j_l$ for one $l \in [1, d]$, one has $k_i^{\mathbf{s}_i}(p_i) = k_j^{\mathbf{s}_j}(p_j)$.

If the pattern of substitution $\mathbf{s} \in \mathcal{S}^{\mathbb{U}_k}$ is compatible with a pattern $p \in \mathcal{A}^{\mathbb{U}_k}$ that appears in a configuration x , it acts on p and we obtain the pattern $\mathbf{s}(p)$

- whose support is $\text{supp}(\mathbf{s}(p)) = \bigcup_{i \in \text{supp}(p)} \mathbb{U}_{\mathbf{k}^{\mathbf{s}_i}(p_i)} + \phi^x(\mathbf{i})$
- and such that $\forall \mathbf{i} \in \text{supp}(p), \forall \mathbf{j} \in \text{supp}(\mathbf{s}_i(p_i)), \mathbf{s}(p)_{\phi^x(\mathbf{i})+\mathbf{j}} = \mathbf{s}_i(p_i)_{\mathbf{j}}$.

Example 3. Let $\mathcal{S} = \{s_a, s_b, s_c, s_d\}$ be a set of two-dimensional substitutions on the alphabet $\mathcal{A} = \{\circ, \bullet\}$ defined by the following rules

$$s_a : \circ \mapsto \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \quad \text{and} \quad \bullet \mapsto \begin{array}{cc} \circ & \circ \\ \bullet & \circ \end{array}, \quad s_b : \circ \mapsto \begin{array}{cc} \circ & \bullet & \circ \\ \circ & \bullet & \circ \end{array} \quad \text{and} \quad \bullet \mapsto \begin{array}{cc} \circ & \circ & \circ \\ \bullet & \bullet & \circ \end{array}$$

$$s_c : \circ \mapsto \begin{array}{cc} \circ & \circ \\ \bullet & \circ \\ \circ & \bullet \end{array} \text{ and } \bullet \mapsto \begin{array}{cc} \circ & \circ \\ \circ & \bullet \\ \bullet & \bullet \end{array}, s_d : \circ \mapsto \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \circ & \circ & \circ \end{array} \text{ and } \bullet \mapsto \begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \bullet & \bullet & \bullet \end{array}.$$

Then given the pattern p pictured below, the patterns of substitution \mathbf{s} and \mathbf{s}' are compatible with p and we can define the patterns $\mathbf{s}(p)$ and $\mathbf{s}'(p)$, while the pattern of substitution \mathbf{s}'' is not.

$$p = \begin{array}{cccc} \circ & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \end{array}, \mathbf{s} = \begin{array}{cccc} s_a & s_a & s_b & s_a \\ s_c & s_c & s_d & s_c \end{array}, \mathbf{s}' = \begin{array}{cccc} s_a & s_a & s_a & s_a \\ s_c & s_c & s_c & s_c \end{array}, \mathbf{s}'' = \begin{array}{cccc} s_a & s_a & s_b & s_a \\ s_c & s_c & s_d & s_a \end{array}$$

$$\mathbf{s}(p) = \begin{array}{|c|c|c|c|} \hline \circ & \circ & \circ & \circ \\ \hline \circ & \circ & \bullet & \bullet \\ \hline \circ & \circ & \bullet & \bullet \\ \hline \circ & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \circ & \circ \\ \hline \end{array}, \mathbf{s}'(p) = \begin{array}{|c|c|c|c|} \hline \circ & \circ & \circ & \circ \\ \hline \bullet & \circ & \circ & \circ \\ \hline \circ & \bullet & \bullet & \bullet \\ \hline \circ & \circ & \circ & \bullet \\ \hline \bullet & \circ & \circ & \circ \\ \hline \end{array}$$

We define the set of \mathcal{S} -patterns by induction. A \mathcal{S} -pattern of level 0 is an element of \mathcal{A} , and p is a \mathcal{S} -pattern of level $n+1$ if there exists a \mathcal{S} -pattern $p' \in \mathcal{A}^{\mathbb{U}}$ of level n and a pattern of substitution $\mathbf{s} \in \mathcal{S}^{\mathbb{U}}$ compatible with p' such that $\mathbf{s}(p') = p$. Of course the support of an \mathcal{S} -pattern is rectangular. The \mathcal{S} -patterns lead us to define $\mathbf{T}_{\mathcal{S}}$, the *local subshift generated by the set of substitutions \mathcal{S}*

$$\mathbf{T}_{\mathcal{S}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall p \sqsubset x, p \text{ is a sub-pattern of a } \mathcal{S}\text{-pattern} \right\}.$$

Suppose that $\mathbf{s} \in \mathcal{S}^{\mathbb{Z}^d}$ is an infinite pattern of substitutions and $x \in \mathcal{A}^{\mathbb{Z}^d}$ is a configuration. We denote by $\mathbf{s}(x)$ the configuration in $\mathcal{A}^{\mathbb{Z}^d}$ obtained if one applies s_i on x_i for every $i \in \mathbb{Z}^d$. We thus define $\mathbf{T}'_{\mathcal{S}}$ the *global subshift generated by the set of substitutions \mathcal{S}* as follows

$$\mathbf{T}'_{\mathcal{S}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall n \in \mathbb{N}, \exists y \in \mathcal{A}^{\mathbb{Z}^d}, \exists (\mathbf{s}_0, \dots, \mathbf{s}_{n-1}) \in (\mathcal{S}^{\mathbb{Z}^d})^n, \right. \\ \left. \mathbf{s}_0 \circ \dots \circ \mathbf{s}_{n-1}(y) = x \right\}.$$

Remark 1. One has both $\mathbf{T}_{\mathcal{S}} \subseteq \mathbf{T}_{\mathbf{S}}$ and $\mathbf{T}'_{\mathbf{S}} \subseteq \mathbf{T}'_{\mathcal{S}}$ for any sequence \mathbf{S} .

3 Realization by sofic subshifts

3.1 Mozes theorem and the property A

In [Moz89] Mozes studied non deterministic multidimensional substitutions, and proved that provided a non deterministic substitution satisfies a good property, then the subshift it generates is sofic.

All substitutions we consider here are deterministic since the substitutions rules are given by a function. Nevertheless this formalism provides a way to study non deterministic substitutions. Given s a non deterministic substitution, if a letter $a \in \mathcal{A}$ has two patterns p_1, p_2 as images, one replaces s by s_1 and s_2 , where s_1 has the same substitutions rules as s without the rule $a \rightarrow p_2$, and

s_2 has the same substitutions rules as s without the rule $a \rightarrow p_1$. By iterating this process, we can transform a non deterministic substitution into a set \mathcal{S} of deterministic substitutions, so that the subshift called (Ω, \mathbb{Z}^2) by Mozes is exactly the subshift $\mathbf{T}_{\mathcal{S}}$.

Theorem 2 ([Moz89]). *Let \mathcal{S} be a set of non degenerate deterministic multidimensional substitutions of type A. Then the subshift $\mathbf{T}_{\mathcal{S}}$ is a sofic.*

We say that a set of substitutions \mathcal{S} is of *type A*, or *has the property A*, if it satisfies the following condition. Let $p = \mathbf{u}_1 \circ \dots \circ \mathbf{u}_k(a)$ be a \mathcal{S} -pattern p and l a 2×2 pattern that appears in p . Suppose there exists a sequence of patterns of substitutions $\mathbf{s}_1, \dots, \mathbf{s}_n$ compatible with the 2×2 pattern l that produce a sequence of patterns $l_0 = l, l_1 = \mathbf{s}_1(l_0), \dots, l_n = \mathbf{s}_n(l_{n-1})$. Then it is possible to find a sequence of patterns of substitution $\mathbf{s}'_1, \dots, \mathbf{s}'_n$ compatible with the pattern p such that the blocks that derive from l in $p_0 = p, p_1 = \mathbf{s}'_1(p_0), \dots, p_n = \mathbf{s}'_n(p_{n-1})$ are exactly l_0, l_1, \dots, l_n (see Figure 2).

Remark 2. This property A for sets of substitutions is actually not very restrictive. For instance any set of substitutions \mathcal{S} such that for every substitution $s \in \mathcal{S}$, the support of $s(a)$ is the same for any $a \in \mathcal{A}$, has the property A. Moreover, if the set \mathcal{S} is reduced to a single deterministic substitution s , then \mathcal{S} is of type A.

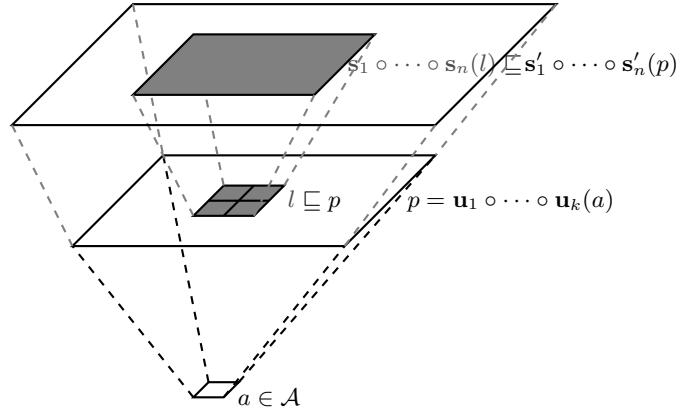


Fig. 2. A set of substitutions \mathcal{S} having the property A.

It is possible that the composition of substitutions rules chosen for l is not compatible with the pattern p , and in this case it is possible to find another

sequence of substitution rules compatible with p and such that the blocks that derive from l are exactly the $l_0 = l, l_1, \dots, l_n$.

Actually one can easily adapt Mozes proof to get a similar result for the subshift $\mathbf{T}'_{\mathcal{S}}$. Moreover we do not require the set of substitutions \mathcal{S} to have the property A .

Corollary 1. *Let \mathcal{S} be a set of deterministic multidimensional substitutions. Then the subshift $\mathbf{T}'_{\mathcal{S}}$ is sofic.*

Idea of the proof. We here give some ideas to adapt of the proof of Theorem 2 to obtain a sketch of the proof of Corollary 1. Let \mathcal{S} be a set of substitutions of type A . Mozes constructs a sofic subshift Σ such that $\mathbf{T}_{\mathcal{S}}$ is a factor of Σ . The subshift Σ contains a grid that ensures that a configuration x is in the sofic subshift Σ if and only if one can find, for any $n \in \mathbb{N}$, a sequence of infinite patterns of substitutions $\mathbf{s}_0, \dots, \mathbf{s}_{n-1} \in \mathcal{S}^{\mathbb{Z}^d}$ and a configuration y_n such that $\mathbf{s}_0 \circ \dots \circ \mathbf{s}_{n-1}(y_n) = x$. In Σ all the y_n are coded in a hierarchical structure. Let Q be the set of 2×2 patterns that appear in a \mathcal{S} -pattern. There is an additional condition : any 2×2 pattern that appears in any configuration y_n is in Q . This construction works: given a configuration $x \in \mathbf{T}_{\mathcal{S}}$ it is easy to construct a $y \in \Sigma$ that encodes x and all its pre-images. Reciprocally given a pattern p that appears in a configuration $x \in \Sigma$, one can find a sequence of finite patterns of substitutions $(\mathbf{s}_0, \dots, \mathbf{s}_{n-1})$ such that p appears in $\mathbf{s}_0 \circ \dots \circ \mathbf{s}_{n-1}(p')$, where p' is either a letter or a 2×1 , a 1×2 or a 2×2 pattern. If p' is a letter then p appears in a \mathcal{S} -pattern. Otherwise, p' appears in a 2×2 pattern that appears itself in a \mathcal{S} -pattern – thanks to condition Q –, hence the property A ensures that p also appears in a \mathcal{S} -pattern, that is to say generated by one letter a (see Figure 2). So any pattern appearing in x appears in a \mathcal{S} -pattern.

So both type A condition and Q condition are made to ensure that any pattern that appears in a configuration x also appears in a \mathcal{S} -pattern.

The difference between the subshifts $\mathbf{T}_{\mathcal{S}}$ and $\mathbf{T}'_{\mathcal{S}}$ is that we remove the condition that forces a pattern appearing in a configuration x to be a \mathcal{S} -pattern – and of course we still require that x has a pre-image of any order by \mathcal{S} . Hence the condition of having the property A is no longer needed, and if we adapt Mozes construction with replacing the set Q by the set of all the 2×2 patterns, then $\mathbf{T}'_{\mathcal{S}}$ is a factor of the sofic subshift obtained. This proves the corollary.

3.2 Effective S-adic subshifts are sofic

Let $\mathbf{S} \in \mathcal{S}^{\mathbb{N}}$, of course one has $\mathbf{T}_{\mathbf{S}} \subset \mathbf{T}_{\mathcal{S}}$, but there is no immediate reason for $\mathbf{T}_{\mathbf{S}}$ to be also sofic, and moreover for cardinality reasons there exists non-sofic S-adic subshifts.

Theorem 3. *Let \mathcal{S} be a finite set of non degenerate multidimensional substitutions and $\mathbf{S} \in \mathcal{S}^{\mathbb{N}}$ be an effective sequence. Then $\mathbf{T}'_{\mathbf{S}}$ is sofic. If \mathcal{S} has the property A , then $\mathbf{T}_{\mathbf{S}}$ is sofic.*

Remark 3. We only present the proof that \mathbf{T}_S is sofic. The proof is similar for \mathbf{T}'_S , one just need to replace \mathbf{T}_S by \mathbf{T}'_S .

Proof. We now assume that $d = 2$, the proof is similar for $d \geq 3$. Let \mathcal{S} be a finite set of non degenerate $(\mathcal{A}, 2)$ -substitutions, we define $\mathcal{A}' = \mathcal{A} \times \mathcal{S}^2$. To every $s \in \mathcal{S}$ we associate a $(\mathcal{A}', 2)$ -substitution \tilde{s} with same support such that

$$\tilde{s} : (a, s_V, s_H) \mapsto \begin{array}{c|cc} \begin{array}{c} (s(a)_{(0, \mathbf{k}_2^s(a))}, s, s_H) \\ (s(a)_{(0, \mathbf{k}_2^s(a)-1)}, s, s) \\ \vdots \\ (s(a)_{(0,1)}, s, s) \\ (s(a)_{(0,0)}, s, s) \end{array} & \begin{array}{c} (s(a)_{(1, \mathbf{k}_2^s(a))}, s, s_H) \quad \dots \quad s(a)_{(\mathbf{k}_1^s(a)-1, \mathbf{k}_2^s(a))}, s, s_H) \\ (s(a)_{(i,j)}, s, s) \\ (s(a)_{(1,0)}, s, s) \quad \dots \quad (s(a)_{(\mathbf{k}_1^s(a)-1,0)}, s, s) \end{array} & \begin{array}{c} (s(a)_{(\mathbf{k}_1^s(a), \mathbf{k}_2^s(a))}, s_V, s_H) \\ (s(a)_{(\mathbf{k}_1^s(a), \mathbf{k}_2^s(a)-1)}, s_V, s) \\ \vdots \\ (s(a)_{(\mathbf{k}_1^s(a),1)}, s_V, s) \\ (s(a)_{(\mathbf{k}_1^s(a),0)}, s_V, s) \end{array} \end{array}$$

All these substitutions \tilde{s} form a set $\tilde{\mathcal{S}} = \{\tilde{s} : s \in \mathcal{S}\}$. Let $\mathbf{S} = (s_i)_{i \in \mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ be an effective sequence, we can thus consider the effective sequence $\tilde{\mathbf{S}} = (\tilde{s}_i)_{i \in \mathbb{N}} \in \tilde{\mathcal{S}}^{\mathbb{N}}$. The aim of substitutions \tilde{s} is to keep a record of the sequence of substitutions previously applied.

Example 4. Let \mathcal{S} be the set of 2-dimensional substitutions on the alphabet $\mathcal{A} = \{\circ, \bullet\}$ defined in Example 3. On the following picture where $\tilde{\mathbf{S}} = (\tilde{s}_d, \tilde{s}_d, \tilde{s}_a, \tilde{s}_a, \dots)$ applied on the letter \bullet , one can find on the bottom line of the patterns $s_2(\bullet), s_1 \circ s_2(\bullet), s_0 \circ s_1 \circ s_2(\bullet), \dots$ the sequence of the substitutions already applied appears.

$$\begin{aligned} (\bullet, s_3, s_3) &\xrightarrow{\tilde{s}_2} \begin{pmatrix} \circ, s_2, s_3 \\ \bullet, s_2, s_2 \end{pmatrix} \begin{pmatrix} \circ, s_3, s_3 \\ \circ, s_3, s_2 \end{pmatrix} \xrightarrow{\tilde{s}_1} \begin{pmatrix} \circ, s_1, s_3 \\ \circ, s_1, s_1 \end{pmatrix} \begin{pmatrix} \circ, s_2, s_3 \\ \circ, s_2, s_1 \end{pmatrix} \begin{pmatrix} \circ, s_1, s_3 \\ \circ, s_1, s_1 \end{pmatrix} \begin{pmatrix} \circ, s_3, s_3 \\ \circ, s_3, s_1 \end{pmatrix} \\ &\quad \begin{pmatrix} \circ, s_1, s_2 \\ \circ, s_1, s_1 \end{pmatrix} \begin{pmatrix} \circ, s_2, s_2 \\ \circ, s_2, s_1 \end{pmatrix} \begin{pmatrix} \circ, s_1, s_2 \\ \circ, s_1, s_1 \end{pmatrix} \begin{pmatrix} \circ, s_3, s_2 \\ \circ, s_3, s_1 \end{pmatrix} \end{aligned}$$

$$\xrightarrow{\tilde{s}_0} \begin{array}{cccccccc} \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_3 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_2, s_3 \\ \bullet, s_2, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_3 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_3 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_3, s_3 \\ \bullet, s_3, s_0 \end{pmatrix} \\ \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_1 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_2, s_1 \\ \bullet, s_2, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_1 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_1 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_3, s_1 \\ \bullet, s_3, s_0 \end{pmatrix} \\ \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_2 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_2, s_2 \\ \bullet, s_2, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_2 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_2 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_3, s_2 \\ \bullet, s_3, s_0 \end{pmatrix} \\ \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_0 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_2, s_0 \\ \bullet, s_2, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_0 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_3, s_0 \\ \bullet, s_3, s_0 \end{pmatrix} \\ \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_0 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_2, s_0 \\ \bullet, s_2, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_1, s_0 \\ \bullet, s_1, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_0, s_0 \\ \bullet, s_0, s_0 \end{pmatrix} & \begin{pmatrix} \bullet, s_3, s_0 \\ \bullet, s_3, s_0 \end{pmatrix} \end{array}$$

One considers $\pi : \mathcal{A}' \rightarrow \mathcal{A}$ the one-block map which keeps the letter of \mathcal{A} and $\pi_V : \mathcal{A}' \rightarrow \mathcal{S}$ (resp. $\pi_H : \mathcal{A}' \rightarrow \mathcal{S}$) the one-block map which keeps the substitution $s_V \in \mathcal{S}$ (resp. $s_H \in \mathcal{S}$) of an element $(a, s_V, s_H) \in \mathcal{A}'$.

CLAIM 1: $\mathbf{T}_S = \pi(\mathbf{T}_{\tilde{S}})$

Proof: This is straightforward, since the alphabet \mathcal{A}' contains alphabet \mathcal{A} , and substitution \tilde{s} restricted to alphabet \mathcal{A} is exactly substitution s . \diamond CLAIM 1

Consequently, it is sufficient to prove that $\mathbf{T}_{\tilde{S}}$ is sofic.

CLAIM 2: The subshift $\Sigma = \mathbf{SA}_{(1,0,\dots,0)\mathbb{Z}}(\pi_V(\mathbf{T}_{\tilde{S}}))$ is effective.

Proof: The class of effective subshifts is closed under factor, but also under projective subaction. This follows from the fact that projective subactions are special cases of factors of subactions, and by Theorem 3.1 and Proposition 3.3 of [Hoc09] which establish that symbolic factors and subactions preserve effectiveness. That is to say $\mathcal{Cl}_{\mathbf{SA}}(\mathcal{RE}) = \mathcal{RE}$. Thus it is sufficient to prove that $\mathbf{T}_{\tilde{\mathbf{S}}}$ is an effective subshift. The sequence of substitutions \mathbf{S} is effective so it is the same for the sequence $\tilde{\mathbf{S}}$. Consequently, one can design an algorithm that computes the $\tilde{\mathbf{S}}$ -patterns, which proves that the subshift $\mathbf{T}_{\tilde{\mathbf{S}}}$ is effective. \diamond Claim 2

By Theorem 1 there exists a d -dimensional subshift of finite type \mathbf{T}_{Σ} on an alphabet \mathcal{B} and a factor $\pi_{\Sigma} : \mathcal{B}^{\mathbb{Z}^d} \rightarrow \mathcal{S}^{\mathbb{Z}^d}$ such that $\Sigma = \pi_{\Sigma}(\mathbf{SA}_{(1,0,\dots,0)\mathbb{Z}}(\mathbf{T}_{\Sigma}))$. Note that the fact that $d \geq 2$ is crucial here, since the previous statement is not true for $d = 1$.

If we consider a configuration of the subshift $\mathbf{T}_{\mathcal{S}}$ defined in Section 2.4, any substitution that appears may be chosen in the set \mathcal{S} , provided it is still compatible with the configuration. But on a same level, different substitutions may appear, which does not fit with the S-adic subshift definition. To solve this problem we synchronize substitutions so that the same substitution is used on a given level. To do that we need to ensure for any configuration $x \in \mathbf{T}_{\mathcal{S}}$, the same substitution appears in $\pi_V(x)$ on each row (resp. in $\pi_H(x)$ on each column).

We thus define the subshift $\tilde{\mathbf{T}}_{\mathcal{S}}$ on the following way :

$$\tilde{\mathbf{T}}_{\mathcal{S}} = \{x \in \mathbf{T}_{\mathcal{S}} : \forall (i, j) \in \mathbb{Z}^2, \pi_H(x)_{(i,j)} = \pi_H(x)_{(i,j+1)} \text{ and } \pi_V(x)_{(i,j)} = \pi_V(x)_{(i+1,j)}\}.$$

We deduce that

$$\tilde{\mathbf{T}}_{\mathcal{S}} = \bigcup_{\tilde{\mathbf{S}} \in \tilde{\mathcal{S}}^{\mathbb{N}}} \mathbf{T}_{\tilde{\mathbf{S}}} \subset \mathbf{T}_{\mathcal{S}}$$

Finally we consider

$$\mathbf{T}_{\text{Final}} = \left\{ (x, s) \in \tilde{\mathbf{T}}_{\mathcal{S}} \times \mathbf{T}_{\Sigma} : \forall (i, j) \in \mathbb{Z}^2, \pi_V(x)_{(i,j)} = \pi_{\Sigma}(s)_{(i,j)} \right\}.$$

Thanks to Corollary 1, we know that the subshift $\mathbf{T}_{\mathcal{S}}$ is sofic, hence so is $\tilde{\mathbf{T}}_{\mathcal{S}}$. Hence by construction, $\mathbf{T}_{\text{Final}}$ is a sofic subshift. Consider the letter-to-letter factor map $\pi_{\text{Final}} : \mathbf{T}_{\text{Final}} \rightarrow \mathcal{A}'^{\mathbb{Z}^d}$ which keeps the letter of \mathcal{A}' .

CLAIM 3: $\pi_{\text{Final}}(\mathbf{T}_{\text{Final}}) = \mathbf{T}_{\tilde{\mathbf{S}}}$

Proof: Given a configuration $x \in \mathbf{T}_{\tilde{\mathbf{S}}}$, it is easy to construct a corresponding element in $\mathbf{T}_{\text{Final}}$. Reciprocally, suppose you are given a configuration $x_{\text{Final}} \in \mathbf{T}_{\text{Final}}$. Replacing substitutions in \mathcal{S} by composition of two substitutions of \mathcal{S} if necessary, we assume that for all $s \in \mathcal{S}$ and all $a \in \mathcal{A}$, $\mathbf{k}_1^s(a), \mathbf{k}_2^s(a) \geq 2$. First the $\tilde{\mathbf{T}}_{\mathcal{S}}$ part of x_{Final} ensures that $\pi_{\text{Final}}(x_{\text{Final}})$ is an element of one $\mathbf{T}_{\tilde{\mathbf{S}'}}$ for some $\tilde{\mathbf{S}'} \in \tilde{\mathcal{S}}^{\mathbb{N}}$. Secondly the condition that links the $\tilde{\mathbf{T}}_{\mathcal{S}}$ part with the \mathbf{T}_{Σ} part certifies that $\mathbf{S}' = \mathbf{S}$: substitution s_0 is the only one which is repeated at least twice systematically – since $\mathbf{k}_1^s, \mathbf{k}_2^s \geq 2$. If we apply the same reasoning to a pre-image of x_{Final} by s_0 , we can find s_1 and so on. \diamond Claim 3

4 Effective subshifts which are S-adic

We want to find a reciprocal statement to Theorem 3: what can we say about an effective subshift which is S-adic? A set of substitutions \mathcal{S} has *unique derivation* if for every element $x \in \widehat{\mathbf{T}}_{\mathcal{S}} = \bigcup_{S \in \mathcal{S}^{\mathbb{N}}} \mathbf{T}_S$, there exist a unique $s \in \mathcal{S}$, a unique $y \in \mathcal{A}^{\mathbb{Z}^d}$ and a unique $i \in \bigcup_{a \in \mathcal{A}} \mathbb{U}_{\mathbf{k}^s(a)}$ such that $s_{\infty}(y) = \sigma^i(x)$.

Theorem 4. *Let \mathcal{S} be a set of substitutions with unique derivation and let $S \in \mathcal{S}^{\mathbb{N}}$. If the S-adic subshift \mathbf{T}_S is effective (and in particular if \mathbf{T}_S is sofic) then S is effective.*

Proof. Since \mathbf{T}_S is effective, there exists a Turing machine \mathcal{M} that enumerates all its forbidden patterns. We compute the sets $\mathcal{E}_s = \{s(a) : a \in \mathcal{A}\}$, and we try to find which substitution $s \in \mathcal{S}$ is the first of the sequence S . To do that, for every $s \in \mathcal{S}$, we try to partition \mathbb{Z}^d with patterns from one \mathcal{E}_s , so that no pattern enumerated by the machine \mathcal{M} appear. These partitions are made in parallel, and the unique derivation condition ensures that only one of them will work. The calculation stops when all the substitutions but one has been rejected – this always happens – and this substitution is s_0 . Apply again this process to a pre-image of x by s , to get the next substitution s_1 , and so on.

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